An introduction to Construction Schemes

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By ω_1 we denote the first uncountable cardinal.

Assume we want to build a structure of size ω_1.

How can we proceed?

One way is to build our structure recursively by countable aproximations. A frequent feature is that many of this aproximations will be elementary substructures of the final structure.

Other methods for building uncountable structures are the following:

1) Todorcevic's method of walks on ordinals
2) Keisler's absoluteness
3) Magidor and Malitz generalization of Keisler's theorem
4) Morasses
5) Forcing

Construction and capturing schemes provide another method for building uncountable structures.

In this approach, we look at finite substructures (instead of countable ones).

This time, the desired structure is obtained by performing amalgamations of many isomorphic finite structures.

The point is that many of the structural properties of the desired uncountable structure, reduce to problems in amalgamating its finite substructures

Construction and Capturing schemes were introduced by Todorcevic in his paper *A construction scheme for non-separable structures.*

Todorcevic, Lopez and Kalajdzievski have publish more papers on the topic.

Construction Schemes

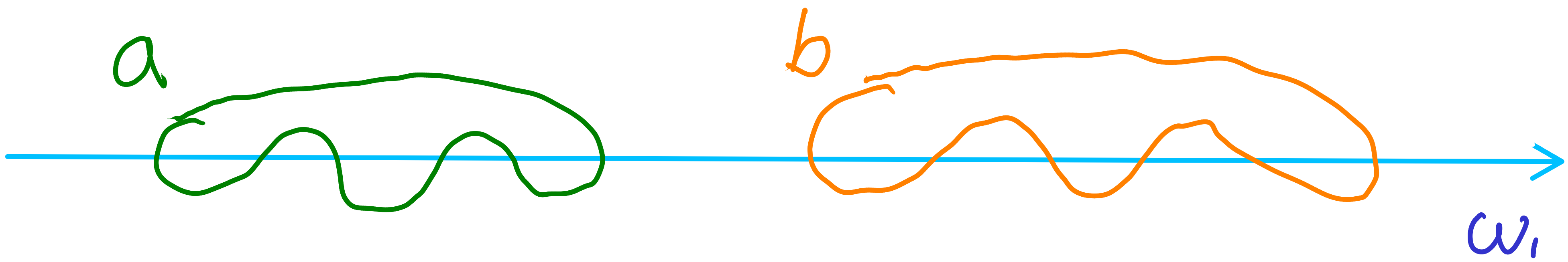
Some notation:

1) $[X]^n = \{s \subseteq X \mid |s| = n\}$

2) $[X]^{<\omega} = \{s \subseteq X \mid |s| < \omega\}$

3) Let $a, b \subseteq \omega_1$. By $a < b$ we mean that:

$$\forall \alpha \in a \ \forall \beta \in b \ (\alpha < \beta)$$



4) Let $a, b \subseteq \omega_1$. $a \sqsubseteq b$ means that a is an initial segment of b. This means:

a) $a \subseteq b$

b) $a < b \setminus a$

Roughly speaking, a construction scheme $\mathcal{F}$ is a family $\mathcal{F} \subseteq [\omega_1]^{<\omega}$ that can be decomposed as $\mathcal{F} = \bigcup_{k \in \omega} \mathcal{F}_k$ such that:

1) $\mathcal{F}_0 = \{\omega_i\}$!

$\mathcal{F}_0$ is the family of all singletons

1) $\mathcal{F}_0 = [\omega_1]^!$

2) $\mathcal{F}$ is cofinal in $[\omega_1]^{<\omega}$.

For every $s \in [\omega_1]^{<\omega}$, there is $F \in \mathcal{F}$ such that $s \subseteq F$

1) $\mathcal{F}_0 = [\omega_1]^1$.

2) $\mathcal{F}$ is cofinal in $[\omega_1]^{<\omega}$.

3) All elements of $\mathcal{F}_k$ have the same size.

If $F, E \in \mathcal{F}_k \Rightarrow |F| = |E|$

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3) All elements of $\mathcal{F}_k$ have the same size.

4) The intersection of two elements of $\mathcal{F}_k$. is an initial segment of both.

If $F, E \in \mathcal{F}_k \Rightarrow F \cap E \sqsubseteq F, E$

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2) $\mathcal{F}$ is cofinal in $[\omega_1]^{<\omega}$.

3) All elements of $\mathcal{F}_k$ have the same size.

4) The intersection of two elements of $\mathcal{F}_k$ is an initial segment of both.

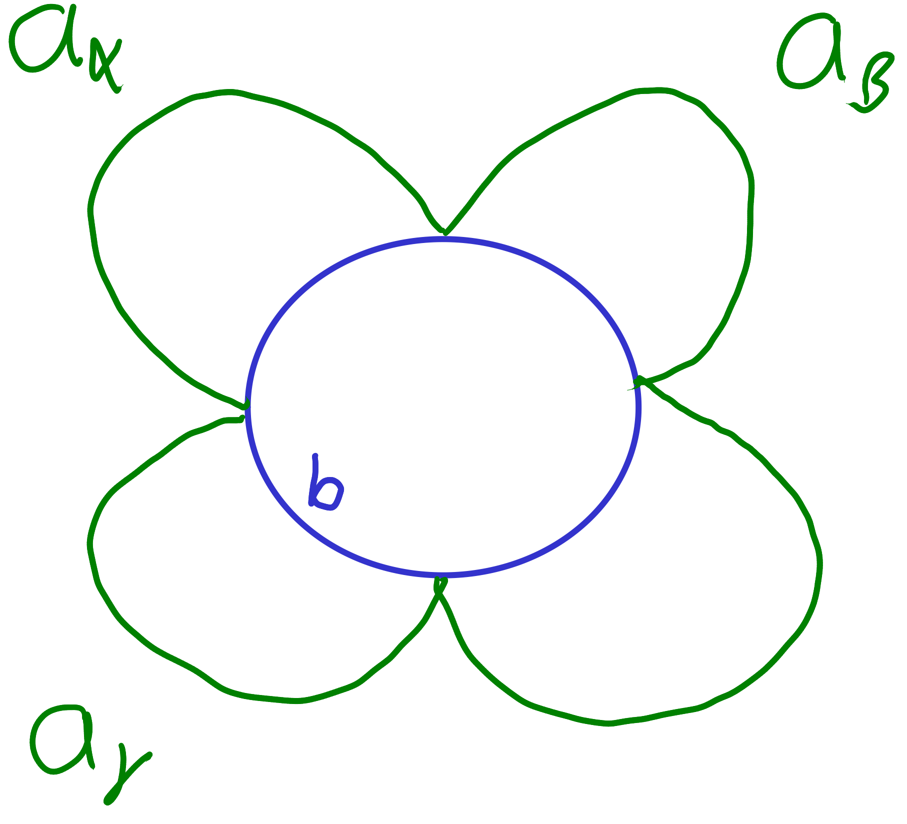
5) Each $F \in \mathcal{F}_{k+1}$ is obtained by amalgamating members of $\mathcal{F}_k$ in a very special way

Def

Let $S = \{a_\alpha \mid \alpha < \kappa\} \subseteq [\omega_1]^{<\omega}$ and $b \subseteq \omega_1$.
We say that S is a Δ-system with root b if:

$$a_\alpha \cap a_\beta = b$$

whenever $\alpha \neq \beta$


a_α
a_β
b
a_γ

Def

We say $\{(m_k, n_{k+1}, r_{k+1})\}_{k\in\omega} \subseteq \omega^3$ is a type if:

1) $m_0 = 1$

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1) $m_0 = 1$

2) $n_k \geq 2$ for $k > 0$

3) $\forall n \in \omega \ \exists^\infty k \in \omega (r_k = n)$

(Will not be relevant for the mini course)

Def

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1) $m_0 = 1$

2) $n_k \geq 2$ for $k>0$

3) $\forall n\in\omega \ \exists^\infty k\in\omega (r_k = n)$

4) $r_{k+1} < m_k$ for all $k\in\omega$.

5) If $k>0$, then:

$$m_{k+1} = r_{k+1} + (m_k - r_{k+1})\, n_{k+1}$$

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$$m_{k+1} = r_{k+1} + (m_k - r_{k+1}) n_{k+1}$$

Do not worry! All conditions have very natural interpretations, as we will see in a moment.

We can now formally state the definition construction scheme.

For this course, $\{(m_{k+1}, n_{k+1}, r_{k+1})\}_{k\in\omega}$ will always denote a type.

$\mathcal{F} \subseteq [\omega_1]^{<\omega}$ is a construction scheme of type $\{(m_{k+1}, n_{k+1}, r_{k+1})\}_{k\in\omega}$ if there is a decomposition $\mathcal{F} = \bigcup_{k\in\omega} \mathcal{F}_k$ such that for all $k\in\omega$:

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2) $\mathcal{F}$ is cofinal in $[\omega_1]^{<\omega}$
3) If $F \in \mathcal{F}_k$, then $|F| = m_k$
4) If $F, E \in \mathcal{F}_k$, then $F \cap E \sqsubseteq F, E$

5) For all $F \in \mathcal{F}_{k+1}$, there are $\{F_i \mid i < n_{k+1}\} \subseteq \mathcal{F}_k$ and $R(F)$ such that:

a) $F = \bigcup_{i < n_{k+1}} F_i$

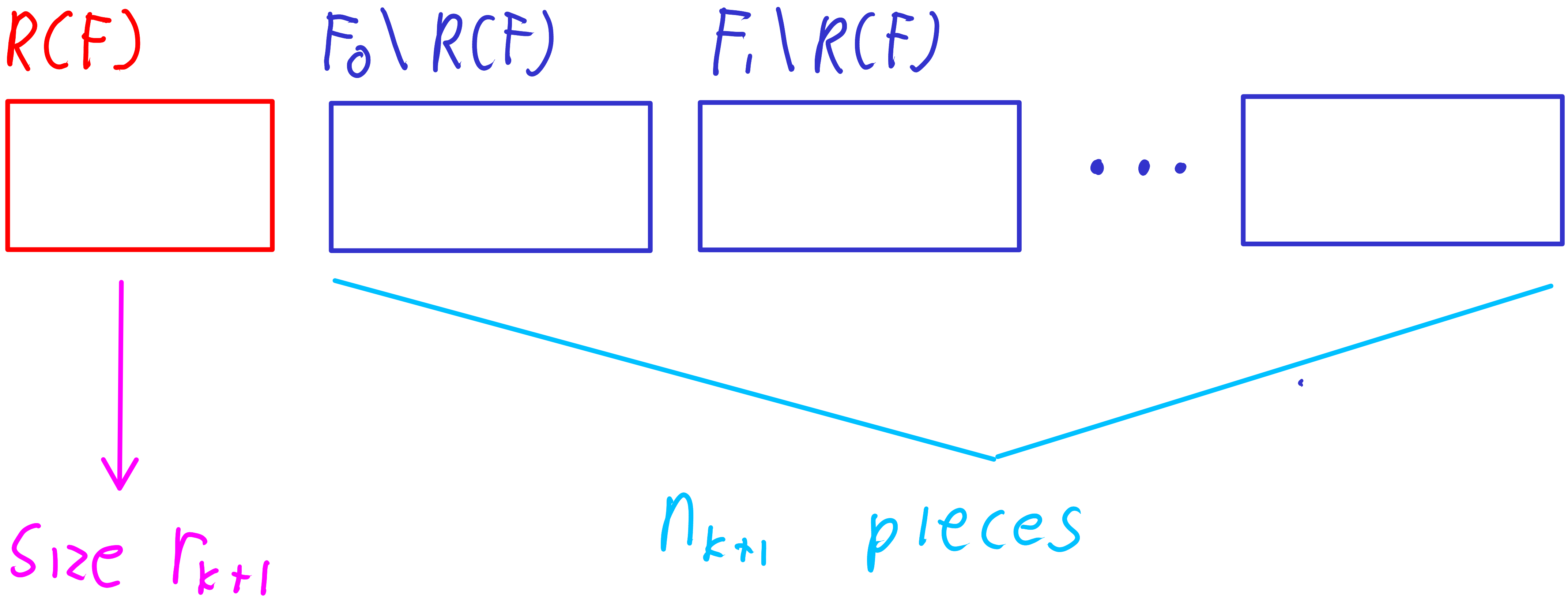
b) $\{F_i \mid i < n_{k+1}\}$ is a Δ-system with root $R(F)$

c) $|R(F)| = r_{k+1}$

d)

$$R(F) < F_0 \setminus R(F) < F_1 \setminus R(F) < \ldots < F_{n_{k+1}-1} \setminus R(F)$$

Each $F \in \mathcal{F}_{k+1}$ looks like this:



Warning!

Every element in $\mathcal{F}_{k+1}$ has this form, but not everything of this form is in $\mathcal{F}_{k+1}$

The Δ-system above $\{F_i \mid i < n_{k+1}\}$ is called the canonical decomposition of F.

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It is not hard to prove that this decomposition is unique

The notion of type is exactly what is needed in order for the previous definition to make sense

m_k n_k r_k

m_k $\qquad n_k$ $\qquad r_k$

Size of the things in F_k

m_k n_k r_k

How many things in $\mathcal{F}_{k-1}$ we glue to build the things in $\mathcal{F}_k$

m_k n_k r_k

The size of the Δ-system

m_k

Size of the things in $\mathcal{F}_k$

n_k

How many things in $\mathcal{F}_{k-1}$ we glue to build the things in $\mathcal{F}_k$

r_k

The size of the Δ-system

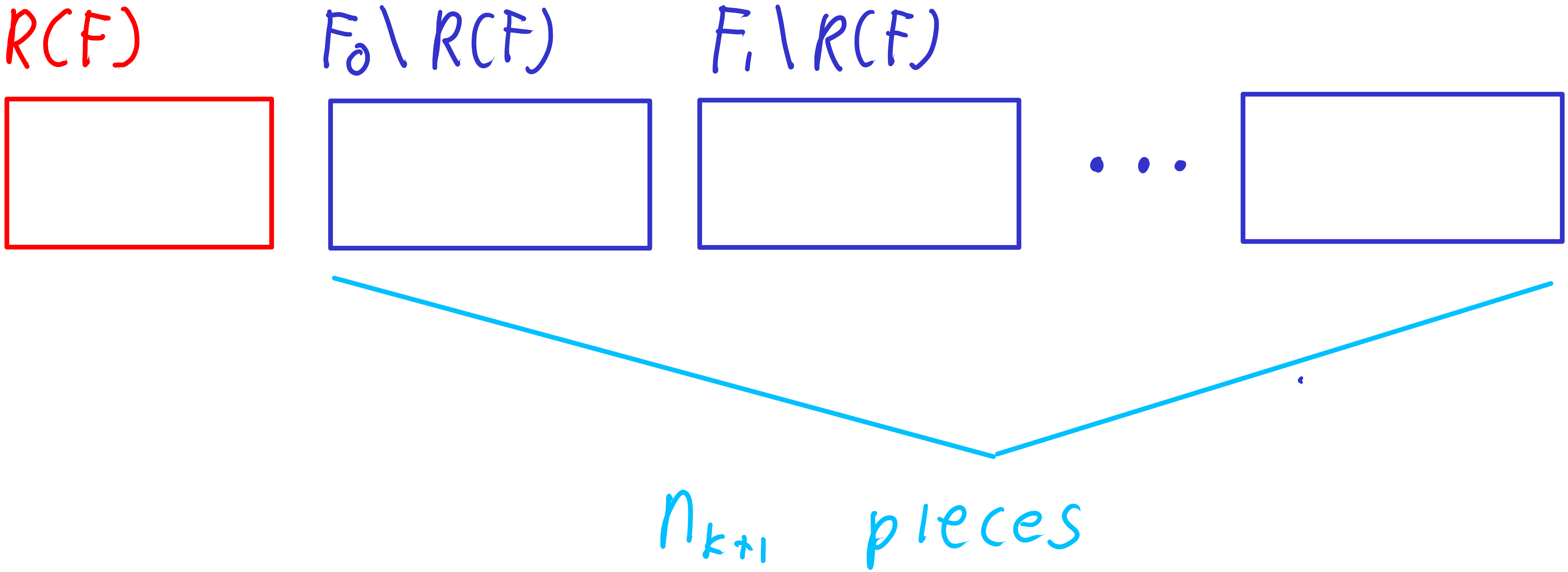
Lets look again at the definition of type.

1) $m_0 = 1$

We want $F_0 = [w_1]'$

2) $n_k \geq 2$ for $k > 0$

We always want to amalgamate at least two things



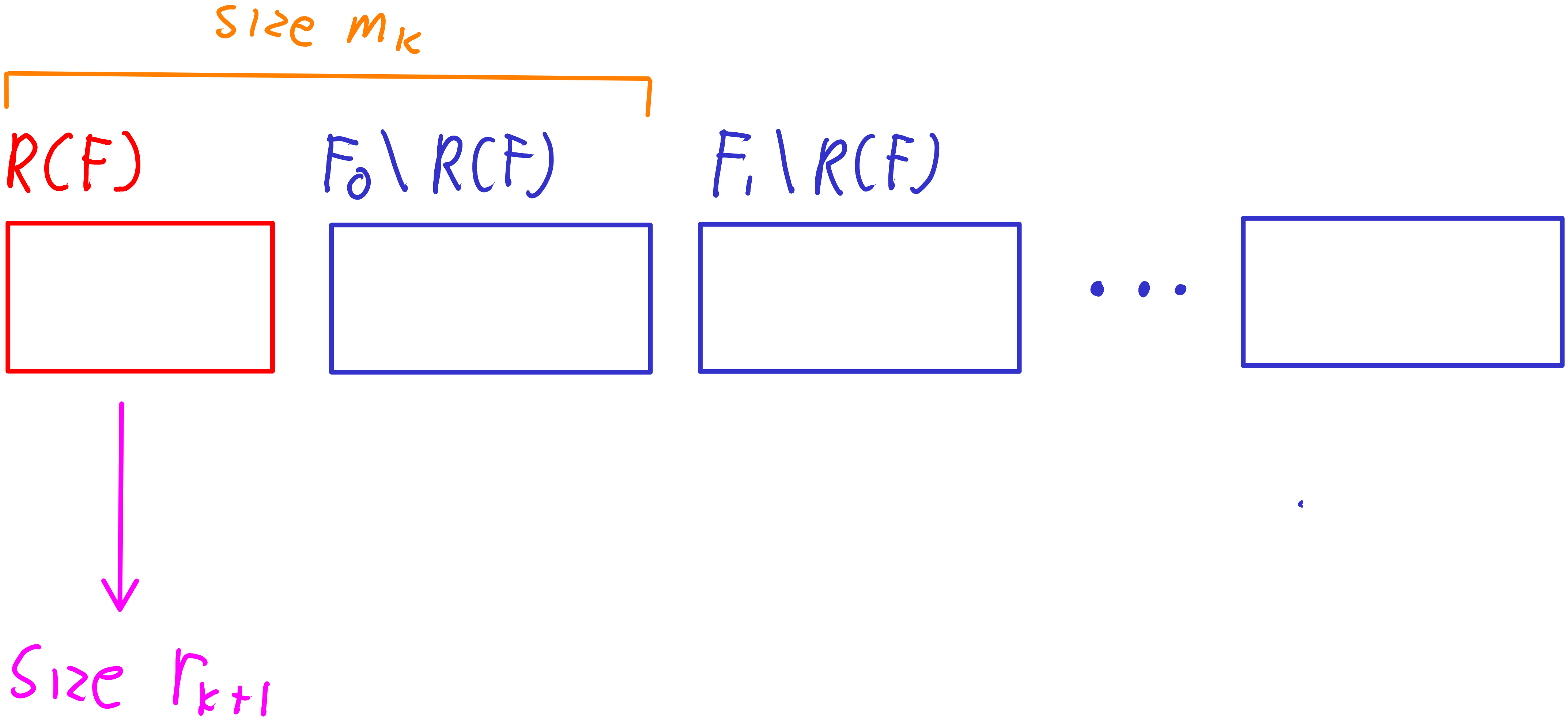
3) $\forall n \in \omega \; \exists^\infty k \in \omega \, (r_k = n)$

Every n appears infinitely many times as the size of a root

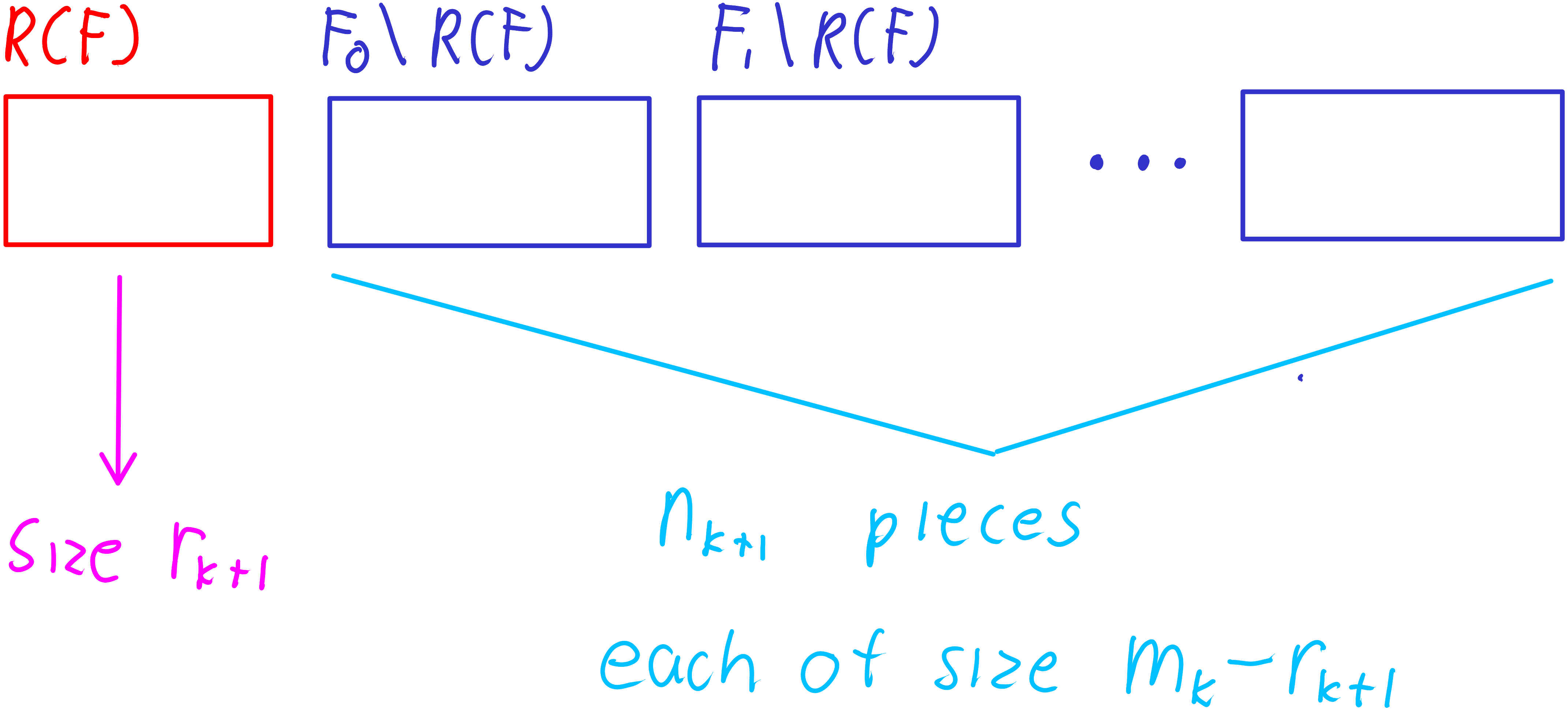
(not relevant for this talks)

4) $r_{k+1} < m_k$ for all $k \in \omega$.

$R(F)$ is subset of F_0



5) $m_{k+1} = r_{k+1} + (m_k - r_{k+1}) n_{k+1}$



Now, the great theorem:

Theorem (Todorcevic)

Construction Schemes exist (for any type)

Some easy properties:

($\mathcal{F}$ will always denote a construction scheme of some type)

Lemma

Let $k<l$. If $F \in \mathcal{F}_k$ and $E \in \mathcal{F}_l$, then $F \cap E \sqsubseteq F$

The intersection is an initial segment of the "smaller one"

Since $k<l$, we get that:

$$E = \bigcup \{ L \in \mathcal{F}_k \mid L \subseteq E \}$$

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so:

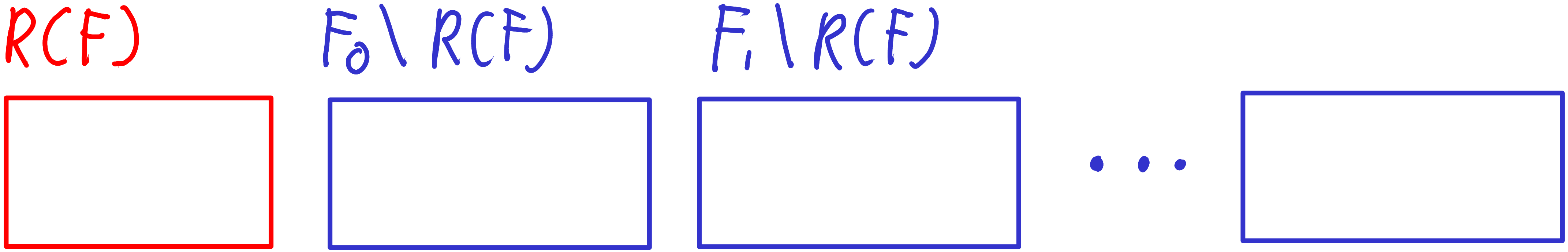
$$E \cap F = \bigcup\{L \cap F \mid L \subseteq E \wedge L \in \mathcal{F}_k\}$$

Each $L \cap F \sqsubseteq F$, so the union is an initial segment of F.

$\square$

It is not in general true that $F \cap E \subseteq E$. Can you think of a counterexample?

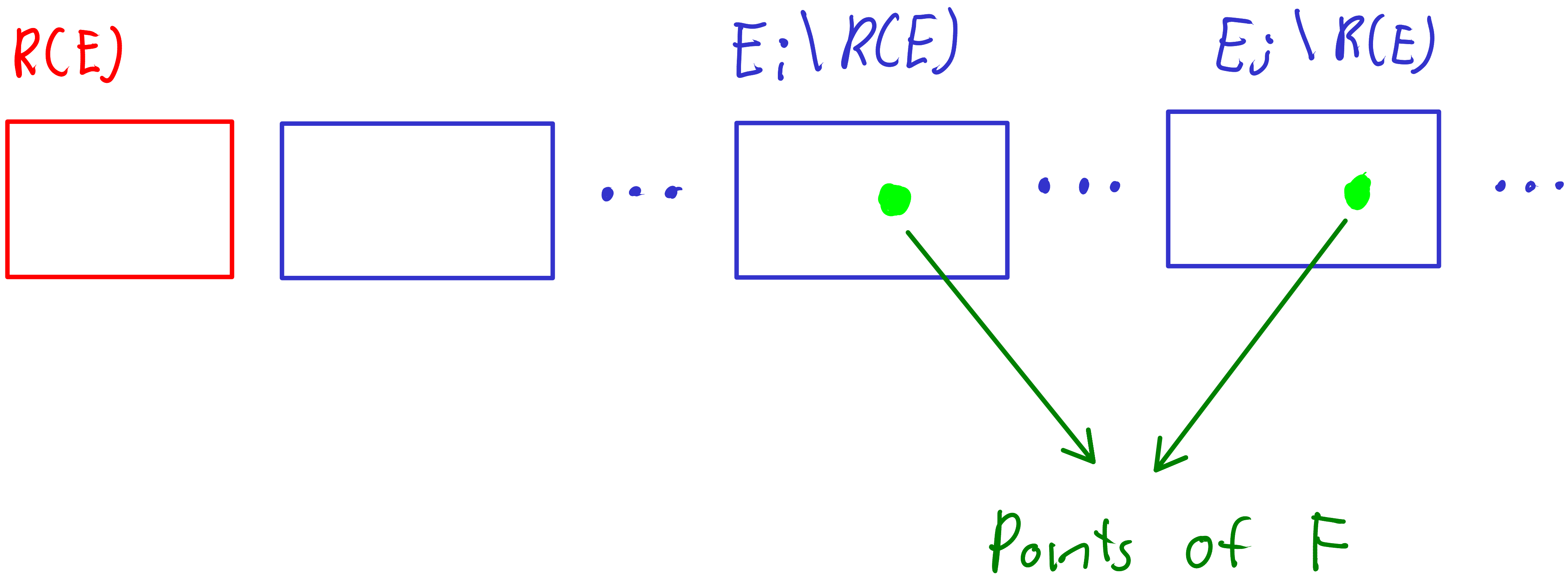
This drawing may help:



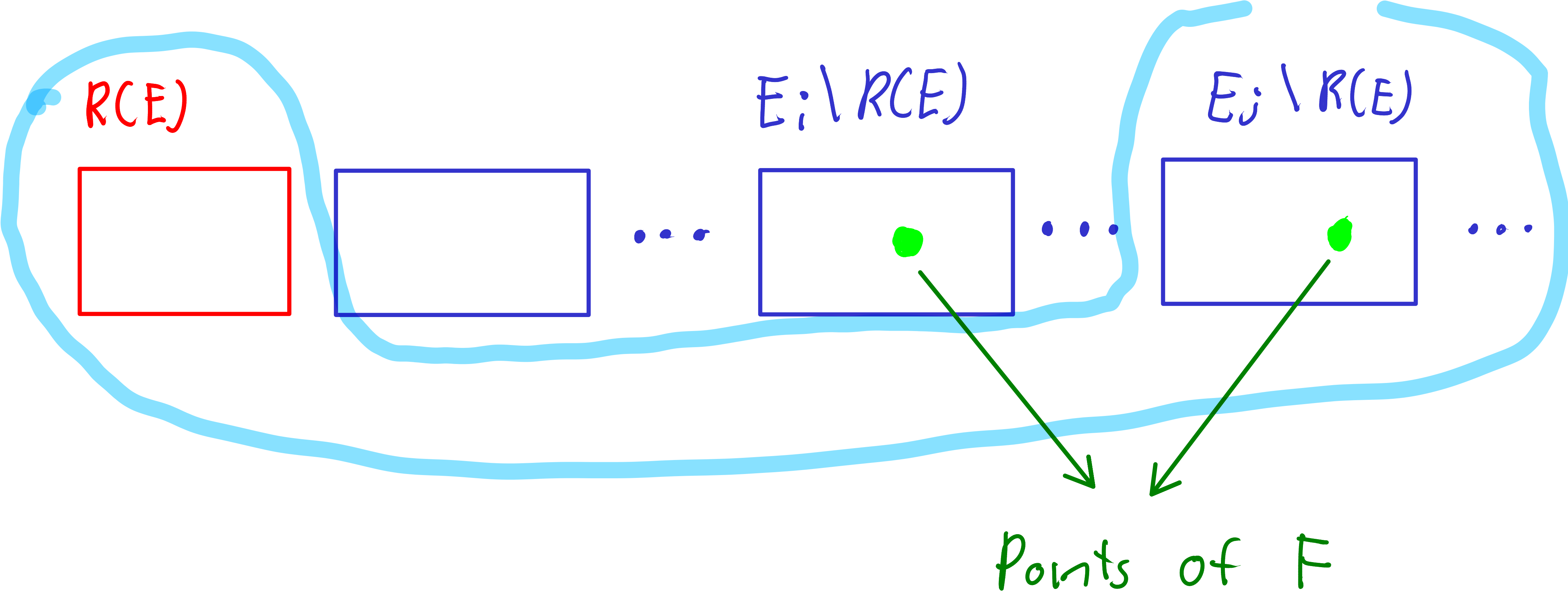
Lemma

Let $k < l$. If $F \in \mathcal{F}_k$, $E \in \mathcal{F}_l$ and $F \subseteq E$, then there is E_i an element of the canonical decomposition E such that $F \subseteq E_i$

Let $\{E_i \mid i < n_\ell\}$ be the canonical decomp. of E. Assume the lemma is false, so there are $i<j$ such that $(E_i \setminus R(E)) \cap F$ and $(E_j \setminus R(E)) \cap F$ are non-empty.



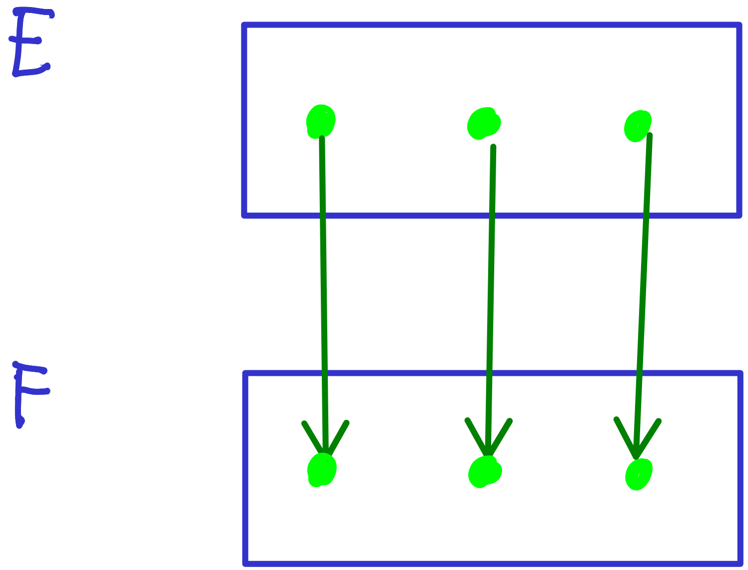
$R(E)$
$E_i \setminus R(E)$
$E_j \setminus R(E)$
Points of F



Then $E_j \cap F$ is not an initial segment of F!

Def

Let $E, F \in [\omega]^{<\omega}$ of the same size. Denote by φ_{EF} the only increasing bijection from E to F.



Lemma

Let $E, F \in \mathcal{F}_k$ and $\varphi = \varphi_{EF}$. Then $\varphi \restriction E \cap F$ is the identity

Lemma

Let $E, F \in \mathcal{F}_k$ and $\varphi = \varphi_{EF}$. Then $\varphi \restriction E \cap F$ is the identity

$E \cap F$ is an initial segment of both E and F.

$\square$

In particular, we get the following:

Lemma

Let $E, F \in \mathcal{F}_k$, $\varphi = \varphi_{EF}$ and $\alpha \in E \cap F$.

Then $\varphi(\alpha) = \alpha$

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Lemma

Let $E, F \in \mathcal{F}_k$, $\varphi = \varphi_{EF}$ and $\alpha \in E \cap F$.
Then $\varphi(\alpha) = \alpha$

This means that α is in the same position in F and in E

(if α is the 283 element of F, then it will be the 283 element of E).

This is a very simple remark, but it is used implicitly all the time.

Def

Let $A \subseteq \omega_1$. Define:

$$\mathcal{F} \upharpoonright A = \{ F \in \mathcal{F} \mid F \subseteq A \}$$

Lemma

Let $F, E \in \mathcal{F}_k$ and $\varphi = \varphi_{FE}$. Then

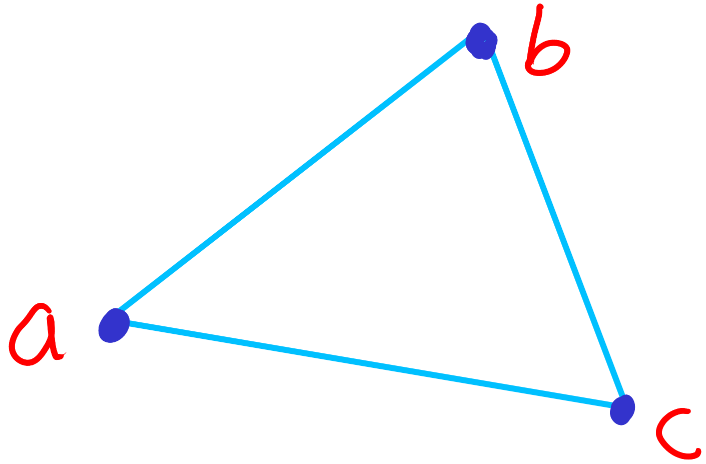
$$\mathcal{F} \restriction E = \{ \varphi[L] \mid L \in \mathcal{F} \restriction F \}$$

Ordinal metrics

Def

Let X be a set. We say that $d: X^2 \longrightarrow \mathbb{R}$ is a metric if for all $a, b, c \in X$:

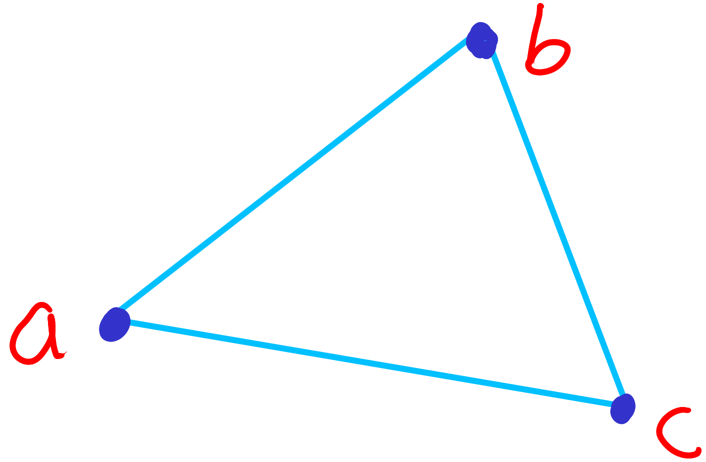
1) $d(a,b) \geq 0$
2) $d(a,b) = 0 \iff a = b$
3) $d(a,b) = d(b,a)$
4) $d(a,b) \leq d(a,c) + d(c,b)$



Def

Let X be a set. We say that $d: X^2 \longrightarrow \mathbb{R}$ is an ultrametric if for all $a, b, c \in X$:

1) $d(a,b) \geq 0$
2) $d(a,b) = 0 \iff a = b$
3) $d(a,b) = d(b,a)$
4) $d(a,b) \leq \max\{d(a,c), d(c,b)\}$

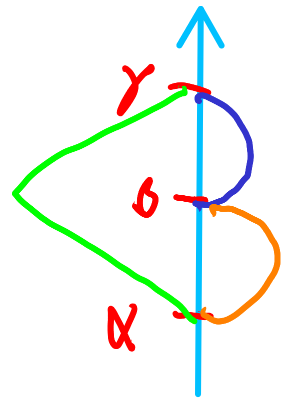


An ordinal metric is a function on ω_1 that resembles an ultrametric, but takes the order on ω_1 into account.

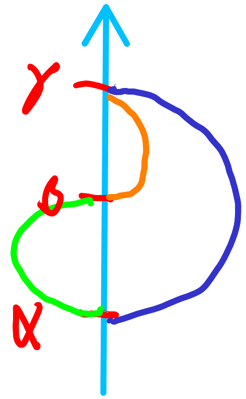
Let $d : \omega_1^2 \longrightarrow \omega$ and $\alpha, \beta, \gamma \in \omega_1$ with $\alpha < \beta < \gamma$.

There are 3 instances of the (ultra) triangle inequality:

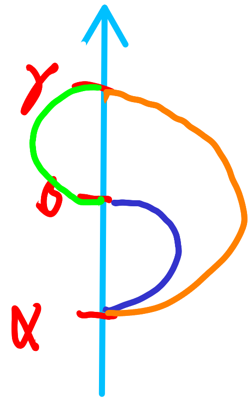
$$d(\alpha,\gamma) \le \max\{d(\alpha,\beta),\ d(\beta,\gamma)\}$$



$$d(\alpha,\beta) \le \max\{d(\alpha,\gamma),\ d(\gamma,\beta)\}$$



$$d(\beta,\gamma) \le \max\{d(\alpha,\beta),\ d(\alpha,\gamma)\}$$



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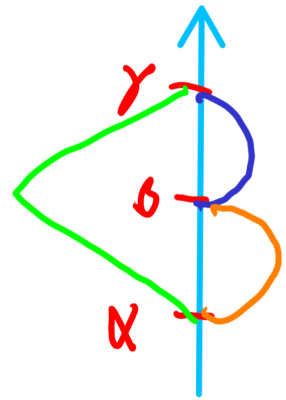
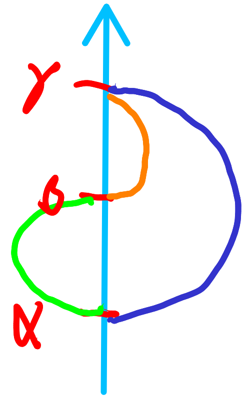
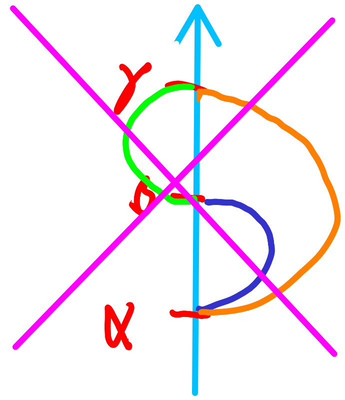


$$d(\alpha,\beta) \le \max\{d(\alpha,\gamma),\ d(\gamma,\beta)\}$$



$$\cancel{d(\beta,\gamma) \le \max\{d(\alpha,\beta),\ d(\alpha,\gamma)\}}$$



Not required for ordinal metrics!

The other component for an ordinal metric is that we required that balls must be finite

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We say that $d : \omega_1^2 \to \omega$ is an ordinal metric if for all $\alpha, \beta, \gamma < \omega_1$:

1) $d(\alpha, \beta) \geq 0$
2) $d(\alpha, \beta) = 0 \iff \alpha = \beta$
3) $d(\alpha, \beta) = d(\beta, \alpha)$

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3) $d(\alpha,\beta) = d(\beta,\alpha)$
4) If $\alpha < \beta < \gamma$, then

$$d(\alpha,\gamma) \leq \max\{d(\alpha,\beta), d(\beta,\gamma)\}$$

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5) For every $l < \omega$, the set:

$$\{\xi \leq \alpha \mid d(\xi, \alpha) \leq l\}$$

is finite

Ordinal metrics were introduced by Todorcevic. His book on walks contains a lot of information about them.

Def

Let $d : \omega_1^2 \longrightarrow \omega$ be an ordinal metric, $\alpha < \omega_1$ and $l \in \omega$. Define:

$$(\alpha)_l = \{ \xi \leq \alpha \mid d(\xi, \alpha) \leq l \}$$

Let $\mathcal{F}$ be a construction scheme.

Define the function:

$$d_{\mathcal{F}} : \omega_1^2 \longrightarrow \omega$$

$$d_{\mathcal{F}}(\alpha, \beta) = \min\{ k \mid \exists F \in \mathcal{F}_k \text{ such that } \{\alpha, \beta\} \subseteq F \}$$

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Define the function:

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(recall $\mathcal{F}$ is cofinal in $[\omega_1]^{<\omega}$, so this is well-defined) prop

1) $d_{\mathcal{F}}$ is an ordinal metric

2) Let $\alpha < \omega_1$ and $k \in \omega$

$$(\alpha)_k = F \cap (\alpha + 1)$$

where F is any $F \in \mathcal{F}_k$ such that $\alpha \in F$

$F \in \mathcal{F}_k$ with $\alpha \in F$